ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT®

J. Math. Anal. Appl. 312 (2005) 357–373

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Existence of solutions to a class of nonlinear second order two-point boundary value problems

Fuyi Li[a,∗], Zhanping Liang[a,b], Qi Zhang[a]

[a] *Department of Mathematics, Shanxi University, Taiyuan 030006, People's Republic of China*
[b] *Department of Mathematics, Xinzhou Teachers' University, Xinzhou 034000, Shanxi, People's Republic of China*

Received 9 January 2005

Available online 13 April 2005

Submitted by J. Henderson

Abstract

In this paper, the existence and multiplicity results of solutions are obtained for the second order two-point boundary value problem $-u''(t) = f(t, u(t))$ for all $t \in [0, 1]$ subject to $u(0) = u'(1) = 0$, where f is continuous. The monotone operator theory and critical point theory are employed to discuss this problem, respectively. In argument, quadratic root operator and its properties play an important role.

Keywords: Two-point boundary value problem; Strongly monotone operator principle; Critical point theory; P.S. condition; Mountain pass lemma

* Corresponding author.
E-mail address: fyli@sxu.edu.cn (F. Li).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.03.043

1. Introduction

In this paper, we consider the existence and multiplicity results of the solutions to the following second order two-point boundary value problem (BVP):

$$\begin{cases} -u''(t) = f(t, u(t)), & t \in [0,1], \\ u(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R}^1 \to \mathbb{R}^1$ is continuous.

Owing to the importance of second order differential equations in physics, the existence and multiplicity of the solutions to the following problem

$$\begin{cases} -u''(t) = f(t, u(t)), & t \in [0,1], \\ u(0) = u(1) = 0 \end{cases}$$

has been studied by many authors, see [1,3–10]. They all obtained the existence results of positive solutions under that f is either superlinear or sublinear in u by employing the cone expansion or compression fixed point theorem. Meanwhile, great importance has been attached to BVP (1.1). But in our knowledge, few papers have discussed the existence results of solutions, especially, infinitely many solutions for BVP (1.1). In this paper, by using the strongly monotone operator principle and the critical point theory, respectively, to discuss BVP (1.1), we establish some conditions for f which are able to guarantee that this problem has a unique solution, at least one nonzero solution, and infinitely many solutions. In argument, $K^{1/2}$, the quadratic root operator of a positive linear compact operator K, and its properties play an important role.

2. Preliminary

In this section, we give some lemmas that are important to our discussion. Let $C[0,1]$ denote the usual real Banach space with the norm $\|u\|_C = \max_{t\in[0,1]} |u(t)|$ for all $u \in C[0,1]$, $L^2[0,1]$ denote the usual real reflexive Banach space with the norm $\|u\| = (\int_0^1 |u(t)|^2\,dt)^{1/2}$ for all $u \in L^2[0,1]$ and the real Hilbert space with the inner product $(u, v) = \int_0^1 u(t)v(t)\,dt$ for all $u, v \in L^2[0,1]$.

It is well known that any solution of BVP (1.1) in $C^2[0,1]$ is equivalent to a solution of the following integral equation in $C[0,1]$,

$$u(t) = \int_0^1 G(t,s) f\big(s, u(s)\big)\,ds, \quad t \in [0,1], \tag{2.1}$$

where $G : [0,1] \times [0,1] \to [0,1]$ is Green's function for $-u''(t) = 0$ for all $t \in [0,1]$ subject to $u(0) = u'(1) = 0$, i.e.,

$$G(t,s) = \min\{t,s\} = \begin{cases} t, & 0 \leqslant t \leqslant s \leqslant 1, \\ s, & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

Define operators $K, \mathbf{f} : C[0,1] \to C[0,1]$ respectively by

$$Ku(t)=\int_0^1 G(t,s)u(s)\,ds,\quad t\in[0,1],\ \forall u\in C[0,1], \tag{2.2}$$

$$\mathbf{f}u(t)=f\big(t,u(t)\big),\quad t\in[0,1],\ \forall u\in C[0,1].$$

Then the integral equation (2.1) is equivalent to the following operator equation:

$$u=K\mathbf{f}u.$$

Remark 2.1. It is easy to see that

(i) $G:[0,1]\times[0,1]\to[0,1]$ is continuous;
(ii) $G(t,s)=G(s,t)$ for all $t,s\in[0,1]$;
(iii) $\max_{(t,s)\in[0,1]\times[0,1]} G(t,s)=1$;
(iv) $\mathbf{f}:C[0,1]\to C[0,1]$ is bounded and continuous.

The operator K defined in (2.2) can also be defined on $L^2[0,1]$. In fact, we have the following lemma.

Lemma 2.1. $K:L^2[0,1]\to C[0,1]$ *is a linear completely continuous operator and then* $K:L^2[0,1]\to L^2[0,1]$ *is also a linear completely continuous operator.*

Proof. For any given $u\in L^2[0,1]$, it follows from (iii) of Remark 2.1 that

$$\big|Ku(t)\big|=\left|\int_0^1 G(t,s)u(s)\,ds\right|\leqslant\int_0^1\big|u(s)\big|\,ds\leqslant\left(\int_0^1\big|u(s)\big|^2\,ds\right)^{1/2}=\|u\|,\quad t\in[0,1]. \tag{2.3}$$

So Ku is a function defined on $[0,1]$. For any given $\varepsilon>0$, since G is continuous on $[0,1]\times[0,1]$, there exists $\delta>0$ such that $|G(t_1,s)-G(t_2,s)|<\varepsilon$ for all t_1,t_2, and s in $[0,1]$ with $|t_1-t_2|<\delta$. And then for all $t_1,t_2\in[0,1]$ with $|t_1-t_2|<\delta$, it follows that

$$\begin{aligned}\big|Ku(t_1)-Ku(t_2)\big|&=\left|\int_0^1\big(G(t_1,s)-G(t_2,s)\big)u(s)\,ds\right|\leqslant\int_0^1\big|G(t_1,s)-G(t_2,s)\big|\big|u(s)\big|\,ds\\&\leqslant\varepsilon\int_0^1\big|u(s)\big|\,ds\leqslant\varepsilon\|u\|.\end{aligned} \tag{2.4}$$

This implies that $Ku\in C[0,1]$. And it follows from (2.3) that

$$\|Ku\|_C\leqslant\|u\|,\quad u\in L^2[0,1]. \tag{2.5}$$

It is obvious that K is linear. So it follows from (2.5) that $K:L^2[0,1]\to C[0,1]$ is continuous.

Let $S \subset L^2[0,1]$ be a bounded subset. Then there exists $M > 0$ such that $\|u\| \leqslant M$ for all $u \in S$. It follows from (2.5) and (2.4) that $K(S)$ is bounded and equicontinuous. According to Arzela–Ascoli theorem, $K(S)$ is a precompact subset of $C[0,1]$. Therefore $K : L^2[0,1] \to C[0,1]$ is completely continuous. Moreover, $K : L^2[0,1] \to L^2[0,1]$ is also completely continuous, since $C[0,1]$ can be continuously embedded into $L^2[0,1]$, which denoted by $C[0,1] \hookrightarrow L^2[0,1]$. The proof is completed. □

Lemma 2.2. $(Ku, v) = (u, Kv)$ *for all* u *and* v *in* $L^2[0,1]$, *i.e.,* K *is symmetric.*

Proof. For any given u and v in $L^2[0,1]$, since

$$\int_0^1 \int_0^1 \left|G(t,s)u(s)v(t)\right| ds\,dt \leqslant \int_0^1 \int_0^1 \left|u(s)v(t)\right| ds\,dt \leqslant \|u\|\|v\|,$$

it follows from Fubini theorem and (ii) of Remark 2.1 that

$$\begin{aligned}(Ku, v) &= \int_0^1 \left[\int_0^1 G(t,s)u(s)\,ds\right] v(t)\,dt = \int_0^1 \int_0^1 G(t,s)u(s)v(t)\,ds\,dt \\ &= \int_0^1 u(s)\left[\int_0^1 G(s,t)v(t)\,dt\right] ds = (u, Kv).\end{aligned}$$

The proof is completed. □

It is easy to see that all eigenvalues of K are $\left\{\frac{4}{(2k-1)^2\pi^2}\right\}_{k=1}^{\infty}$, which have the corresponding orthonormal eigenfunctions $\{e_k\}_{k=1}^{\infty} = \{\sqrt{2}\sin((2k-1)\pi t/2)\}_{k=1}^{\infty}$. Since K is linear completely continuous and symmetric by Lemmas 2.1 and 2.2, the following formulas with K hold:

$$u = \sum_{k=1}^{\infty} (u, e_k)e_k, \quad u \in L^2[0,1], \tag{2.6}$$

$$\|u\|^2 = \sum_{k=1}^{\infty} \left|(u, e_k)\right|^2, \quad u \in L^2[0,1], \tag{2.7}$$

$$Ku = \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2}(u, e_k)e_k, \quad u \in L^2[0,1]. \tag{2.8}$$

Remark 2.2. It follows from Lemmas 2.1, 2.2 and (2.8) that $K : L^2[0,1] \to L^2[0,1]$ is positive linear bounded and symmetric. So $K^{1/2} : L^2[0,1] \to L^2[0,1]$, the positive square root of K, exists and is unique, and is also linear bounded and symmetric with $\|K^{1/2}\| = \|K\|^{1/2}$. Furthermore, we can also prove that $K^{1/2}$ maps $L^2[0,1]$ into $C[0,1]$ and $K^{1/2} : L^2[0,1] \to C[0,1]$ is completely continuous.

Lemma 2.3. *$K^{1/2} : L^2[0,1] \to C[0,1]$ is a linear completely continuous operator. Then $K^{1/2} : L^2[0,1] \to L^2[0,1]$ is also linear completely continuous.*

Proof. From (2.8), $K^{1/2} : L^2[0,1] \to L^2[0,1]$ is of form as follows:

$$K^{1/2}u = \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi}(u, e_k)e_k, \quad u \in L^2[0,1]. \tag{2.9}$$

For any given $u \in L^2[0,1]$, and positive integers n, p, from (2.7) and $|e_k(t)| \leqslant \sqrt{2}$ for all $t \in [0,1]$, $k = 1, 2, \ldots$, we have that

$$\begin{aligned}\left|\sum_{k=n+1}^{n+p} \frac{2}{(2k-1)\pi}(u, e_k)e_k(t)\right| &\leqslant \sum_{k=n+1}^{n+p} \frac{2\sqrt{2}}{(2k-1)\pi}\left|(u, e_k)\right| \\ &\leqslant \left(\sum_{k=n+1}^{n+p} \frac{8}{(2k-1)^2\pi^2}\right)^{1/2} \left(\sum_{k=n+1}^{n+p} \left|(u, e_k)\right|^2\right)^{1/2} \\ &\leqslant \left(\sum_{k=n+1}^{n+p} \frac{8}{(2k-1)^2\pi^2}\right)^{1/2} \|u\|, \quad t \in [0,1].\end{aligned}$$

It follows that

$$\left\|\sum_{k=n+1}^{n+p} \frac{2}{(2k-1)\pi}(u, e_k)e_k\right\|_C \leqslant \left(\sum_{k=n+1}^{n+p} \frac{8}{(2k-1)^2\pi^2}\right)^{1/2} \|u\| \to 0, \quad n \to \infty. \tag{2.10}$$

Hence, the series $\sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi}(u, e_k)e_k(t)$ converges uniformly with respect to $t \in [0,1]$, and then $K^{1/2}u \in C[0,1]$.

In addition, it is easy to see from (2.10) that

$$\|K^{1/2}u\|_C \leqslant \left(\sum_{k=1}^{\infty} \frac{8}{(2k-1)^2\pi^2}\right)^{1/2} \|u\| = \|u\|, \quad u \in L^2[0,1]. \tag{2.11}$$

Taking

$$T_n u = \sum_{k=1}^{n} \frac{2}{(2k-1)\pi}(u, e_k)e_k, \quad u \in L^2[0,1],\ n = 1, 2, \ldots.$$

Then $T_n : L^2[0,1] \to C[0,1]$ is a linear completely continuous operator, $n = 1, 2, \ldots$, and it follows from (2.9) and (2.10) that

$$\|T_n - K^{1/2}\| \leqslant \left(\sum_{k=n+1}^{\infty} \frac{8}{(2k-1)^2\pi^2}\right)^{1/2} \to 0, \quad n \to \infty.$$

Therefore, $K^{1/2} : L^2[0,1] \to C[0,1]$ is a linear completely continuous operator. And then $K^{1/2} : L^2[0,1] \to L^2[0,1]$ is also a linear completely continuous operator since $C[0,1] \hookrightarrow L^2[0,1]$. The proof is completed. □

Remark 2.3. It follows from (2.9) and (2.6) that

$$(K^{1/2}u, u) = \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \left|(u, e_k)\right|^2, \quad u \in L^2[0,1].$$

This and (2.7) imply that $K^{1/2}u \neq 0$ for all $u \in L^2[0,1]$ with $u \neq 0$. Therefore $K^{1/2}u_1 \neq K^{1/2}u_2$ for all $u_1, u_2 \in L^2[0,1]$ with $u_1 \neq u_2$.

Theorem 2.1 [13, Theorem 4.1, p. 354]. *Suppose A is a linear compact, symmetric operator, and $A \neq 0$. Then $\|A\|$ or $-\|A\|$ is an eigenvalue of A.*

Since the all eigenvalues of K are $\{4/((2k-1)^2\pi^2)\}_{k=1}^{\infty}$, and K is linear compact and symmetric by Lemmas 2.1 and 2.2, it follows from Theorem 2.1 that $\|K\| = 4/\pi^2$, where K is from $L^2[0,1]$ into itself. It follows in the same way that $\|K^{1/2}\| = 2/\pi$.

In the next section, we will use the strongly monotone operator principle and the critical point theory, respectively, to discuss BVP (1.1). Here we state some necessary definitions and theorems. For instance, see [2,5,11,12,14,16].

Definition 2.1. Let E be a real Banach space, D an open subset of E. Suppose that a functional $f : D \to \mathbb{R}^1$ is Fréchet differentiable on D. If $x_0 \in D$ and the Fréchet derivative $f'(x_0) = 0$, then we call that x_0 is a critical point of the functional f and $c = f(x_0)$ is a critical value of f.

Let $C^1(E, \mathbb{R}^1)$ denote the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on E.

Definition 2.2. Let $f \in C^1(E, \mathbb{R}^1)$. If any sequence $\{u_n\} \subset E$ for which $\{f(u_n)\}$ is bounded and $f'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence, then we say f satisfies Palais–Smale condition (denoted by P.S. condition for short).

Theorem 2.2 (Strongly monotone operator principle [5, Corollary 2, p. 373]). *Let E be a real reflexive Banach space. Suppose that $T : E \to E^*$ is a continuous operator and there exists $c > 0$ such that*

$$(Tu - Tv, u - v) \geqslant c\|u - v\|^2, \quad u, v \in E.$$

Then $T : E \to E^*$ is a homeomorphism between E and E^*.

Theorem 2.3 [5, Theorem 1.7, p. 423]. *Let E be a real reflexive Banach space. If the functional $f : E \to \mathbb{R}^1$ is weakly lower semi-continuous and satisfies $\lim_{\|x\| \to \infty} f(x) = +\infty$, then there exists $x_0 \in E$ such that $f(x_0) = \inf_{x \in E} f(x)$. Moreover, if f is also Fréchet differentiable on E, then $f'(x_0) = 0$.*

For convenience of readers, we suggest one can also refer to [2,12,14,16] for the content involved in Theorems 2.2 and 2.3.

Theorem 2.4 (Mountain pass lemma [11, Theorem 2.2, p. 7]; [12, Theorem 6.1, p. 109]). *Let E be a real Banach space and $f \in C^1(E, \mathbb{R}^1)$ satisfying P.S. condition. Suppose that*

(i) $f(0) = 0$;
(ii) *there exist $\rho > 0$ and $\alpha > 0$ such that $f(u) \geqslant \alpha$ for all $u \in E$ with $\|u\| = \rho$;*
(iii) *there exists u_1 in E with $\|u_1\| \geqslant \rho$ such that $f(u_1) < \alpha$.*

Then f possesses a critical value $c \geqslant \alpha$. Moreover, c can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} f(u),$$

where $\Gamma = \{g \in C([0,1], E)\colon\ g(0) = 0,\ g(1) = u_1\}$.

Theorem 2.5 ([11, Theorem 9.12, p. 55]; [12, Theorem 6.5, p. 114]). *Let E be an infinite dimensional real Banach space and let $f \in C^1(E, \mathbb{R}^1)$ be even, satisfy P.S. condition, and $f(0) = 0$. Suppose that $E = V \oplus X$, where V is finite dimensional, and f satisfies*

(i) *there exist $\alpha > 0$ and $\rho > 0$ such that $f(u) \geqslant \alpha$ for all $u \in X$ with $\|u\| = \rho$;*
(ii) *for any finite dimensional subspace $W \subset E$, there is $R = R(W)$ such that $f(u) \leqslant 0$ for all $u \in W$ with $\|u\| \geqslant R$.*

Then f possesses an unbounded sequence of critical values.

3. Main results

In this section, we deal with the existence of one solution and infinitely many solutions for BVP (1.1). We will use the strongly monotone operator principle and the critical point theory respectively.

Lemma 3.1.

(i) *The operator equation*

$$u = K\mathbf{f}u \tag{3.1}$$

has a solution in $C[0,1]$ if and only if the operator equation

$$v = K^{1/2}\mathbf{f}K^{1/2}v \tag{3.2}$$

has a solution in $L^2[0,1]$.
(ii) *The uniqueness of solution for these two above equations is also equivalent.*
(iii) *If* (3.2) *has a nonzero solution in $L^2[0,1]$, then* (3.1) *has a nonzero solution in $C[0,1]$. If* (3.2) *has infinitely many solutions in $L^2[0,1]$, then* (3.1) *has infinitely many solutions in $C[0,1]$.*

Proof. (i) Let $u \in C[0,1]$ be any solution of (3.1), i.e., $u = K\mathbf{f}u$. Then $K^{1/2}\mathbf{f}u = K^{1/2}\mathbf{f}K\mathbf{f}u = K^{1/2}\mathbf{f}K^{1/2}K^{1/2}\mathbf{f}u = K^{1/2}\mathbf{f}K^{1/2}(K^{1/2}\mathbf{f}u)$, so $v = K^{1/2}\mathbf{f}u \in C[0,1] \hookrightarrow$

$L^2[0,1]$ is a solution of (3.2). On the other hand, let $v \in L^2[0,1]$ be a solution of (3.2), i.e., $v = K^{1/2}\mathbf{f}K^{1/2}v$. Then $K^{1/2}v = K^{1/2}K^{1/2}\mathbf{f}K^{1/2}v = K\mathbf{f}(K^{1/2}v)$, so $u = K^{1/2}v \in C[0,1]$ is a solution of (3.1).

(ii) Assume that (3.1) has a unique solution u in $C[0,1]$. Let v_1 and v_2 in $L^2[0,1]$ be both solutions of (3.2), i.e., $v_1 = K^{1/2}\mathbf{f}K^{1/2}v_1$ and $v_2 = K^{1/2}\mathbf{f}K^{1/2}v_2$. Then $K^{1/2}v_1 = K\mathbf{f}K^{1/2}v_1$ and $K^{1/2}v_2 = K\mathbf{f}K^{1/2}v_2$. And then $K^{1/2}v_1$ and $K^{1/2}v_2$ are both solutions of (3.1). It follows from the assumption that $K^{1/2}v_1 = K^{1/2}v_2 = u$, so $v_1 = K^{1/2}\mathbf{f}K^{1/2}v_1 = K^{1/2}\mathbf{f}K^{1/2}v_2 = v_2$. On the other hand, assume that (3.2) has a unique solution v in $L^2[0,1]$. Let u_1 and u_2 in $C[0,1]$ be both solutions of (3.1), i.e., $u_1 = K\mathbf{f}u_1$ and $u_2 = K\mathbf{f}u_2$. Then $K^{1/2}\mathbf{f}u_1 = K^{1/2}\mathbf{f}K^{1/2}K^{1/2}\mathbf{f}u_1$ and $K^{1/2}\mathbf{f}u_2 = K^{1/2}\mathbf{f}K^{1/2}K^{1/2}\mathbf{f}u_2$. And then $K^{1/2}\mathbf{f}u_1$ and $K^{1/2}\mathbf{f}u_2$ are both solutions of (3.2). It follows from the assumption that $K^{1/2}\mathbf{f}u_1 = K^{1/2}\mathbf{f}u_2 = v$, so $u_1 = K\mathbf{f}u_1 = K^{1/2}K^{1/2}\mathbf{f}u_1 = K^{1/2}K^{1/2}\mathbf{f}u_2 = K\mathbf{f}u_2 = u_2$.

(iii) It follows from the proof of (i) that if $v \in L^2[0,1]$ is any solution of (3.2), then $K^{1/2}v$ is a solution of (3.1) in $C[0,1]$. Therefore, Remark 2.3 yields the conclusion. The proof is completed. □

Theorem 3.1. *Suppose that for each $t \in [0,1]$, $f(t,u)$ is a nonincreasing function in u, i.e., $f(t,u_1) \geqslant f(t,u_2)$ for all u_1 and u_2 in $\mathbb{R}^1$ with $u_1 < u_2$. Then BVP* (1.1) *has a unique solution in $C^2[0,1]$.*

Proof. It follows from Lemma 3.1 that the operator equation (3.1) has a unique solution in $C[0,1]$ if and only if $v = K^{1/2}\mathbf{f}K^{1/2}v$ has a unique solution in $L^2[0,1]$. Therefore, it is also equivalent to that $Tv = 0$ has a unique solution in $L^2[0,1]$, where $T = I - K^{1/2}\mathbf{f}K^{1/2}$. From Lemma 2.3, $K^{1/2} : L^2[0,1] \to C[0,1] \hookrightarrow L^2[0,1]$ is continuous, so $T : L^2[0,1] \to L^2[0,1]$ is also continuous. According to Theorem 2.2, it is only necessary to verify that T is a strongly monotone operator. In fact, for all u and v in $L^2[0,1]$, since $f(t,u)$ is a nonincreasing function in u for each $t \in [0,1]$, it follows that

$$\begin{aligned}
&(\mathbf{f}K^{1/2}u - \mathbf{f}K^{1/2}v, K^{1/2}u - K^{1/2}v)\\
&\quad = \int_0^1 \bigl[f\bigl(t, K^{1/2}u(t)\bigr) - f\bigl(t, K^{1/2}v(t)\bigr)\bigr]\bigl[K^{1/2}u(t) - K^{1/2}v(t)\bigr]\,dt \leqslant 0.
\end{aligned}$$

And using Remark 2.2, we have that

$$\begin{aligned}
(Tu - Tv, u - v) &= (u - v - K^{1/2}\mathbf{f}K^{1/2}u + K^{1/2}\mathbf{f}K^{1/2}v, u - v)\\
&= \|u - v\|^2 - \bigl(K^{1/2}(\mathbf{f}K^{1/2}u - \mathbf{f}K^{1/2}v), u - v\bigr)\\
&= \|u - v\|^2 - (\mathbf{f}K^{1/2}u - \mathbf{f}K^{1/2}v, K^{1/2}u - K^{1/2}v)\\
&\geqslant \|u - v\|^2, \quad u, v \in L^2[0,1].
\end{aligned}$$

Thus, T is a strongly monotone operator. And then it follows by Theorem 2.2 that $Tv = 0$ has a unique solution v^* in $L^2[0,1]$. The proof is completed. □

Theorem 3.2. *If there exists $a \in [0, \pi^2/4)$ such that $[f(t,u) - f(t,v)][u - v] \leqslant a|u - v|^2$ for all $t \in [0,1]$, and $u, v \in \mathbb{R}^1$, then BVP* (1.1) *has a unique solution in $C^2[0,1]$.*

Proof. By the same way as the proof of Theorem 3.1, it is only necessary to note that

$$\begin{aligned}
&(\mathbf{f}K^{1/2}u - \mathbf{f}K^{1/2}v, K^{1/2}u - K^{1/2}v) \\
&\quad = \int_0^1 \bigl[f\bigl(t, K^{1/2}u(t)\bigr) - f\bigl(t, K^{1/2}v(t)\bigr)\bigr]\bigl[K^{1/2}u(t) - K^{1/2}v(t)\bigr]\,dt \\
&\quad \leqslant a\int_0^1 \bigl|K^{1/2}(u-v)(t)\bigr|^2\,dt \\
&\quad = a\bigl(K^{1/2}(u-v), K^{1/2}(u-v)\bigr) \\
&\quad = a\bigl(K(u-v), u-v\bigr) \\
&\quad \leqslant \frac{4a}{\pi^2}\|u-v\|^2, \quad u, v \in L^2[0,1].
\end{aligned}$$

And then

$$\begin{aligned}
(Tu - Tv, u - v) &= (u - v - K^{1/2}\mathbf{f}K^{1/2}u + K^{1/2}\mathbf{f}K^{1/2}v, u - v) \\
&= \|u-v\|^2 - (\mathbf{f}K^{1/2}u - \mathbf{f}K^{1/2}v, K^{1/2}u - K^{1/2}v) \\
&\geqslant \|u-v\|^2 - \frac{4a}{\pi^2}\|u-v\|^2 \\
&= (1 - 4a/\pi^2)\|u-v\|^2, \quad u, v \in L^2[0,1].
\end{aligned}$$

The proof is completed. □

Lemma 3.2. *Let* $\Phi(u) = \int_0^1 \int_0^{u(t)} f(t, v)\,dv\,dt$, $u \in C[0,1]$. *Then*

(i) $\Phi : C[0,1] \to \mathbb{R}^1$ *is Fréchet differentiable on* $C[0,1]$ *and* $(\Phi'(u))(w) = (\mathbf{f}u, w) = \int_0^1 f(t, u(t))w(t)\,dt$ *for all* $u, w \in C[0,1]$;
(ii) $\Phi \circ K^{1/2} : L^2[0,1] \to \mathbb{R}^1$ *is Fréchet differentiable on* $L^2[0,1]$ *and* $(\Phi \circ K^{1/2})'(v) = K^{1/2}\mathbf{f}K^{1/2}v$ *for all* $v \in L^2[0,1]$.

Proof. (i) For any given $u \in C[0,1]$, we define a bounded linear functional $h(w) = (\mathbf{f}u, w) = \int_0^1 f(t, u(t))w(t)\,dt$ for all $w \in C[0,1]$, then

$$\begin{aligned}
&\Phi(u+w) - \Phi(u) - h(w) \\
&\quad = \int_0^1 \int_{u(t)}^{u(t)+w(t)} f(t, v)\,dv\,dt - \int_0^1 f\bigl(t, u(t)\bigr)w(t)\,dt \\
&\quad = \int_0^1 f\bigl(t, u(t) + \theta w(t)\bigr)w(t)\,dt - \int_0^1 f\bigl(t, u(t)\bigr)w(t)\,dt
\end{aligned}$$

$$= \int_0^1 \left[f\left(t, u(t) + \theta w(t)\right) - f\left(t, u(t)\right)\right] w(t)\, dt, \quad w \in C[0,1],$$

where $\theta \in (0,1)$. Since f is continuous on $[0,1] \times [-\|u\|_C - 1, \|u\|_C + 1]$, we have

$$\lim_{\|w\|_C \to 0} \frac{1}{\|w\|_C} \int_0^1 \left[f\left(t, u(t) + \theta w(t)\right) - f\left(t, u(t)\right)\right] w(t)\, dt = 0.$$

Then, Φ is Fréchet differentiable on $C[0,1]$ and $(\Phi'(u))(w) = h(w) = (\mathbf{f}u, w)$ for all $u, w \in C[0,1]$.

(ii) By the chain rule for derivatives of composite operator [15, Proposition 4.10, p. 139], it follows that the functional $\Phi \circ K^{1/2}$ is Fréchet differentiable on $L^2[0,1]$ and $(\Phi \circ K^{1/2})'(v) = \Phi'(K^{1/2}v) \circ K^{1/2}$ for all $v \in L^2[0,1]$, that is, $((\Phi \circ K^{1/2})'(v))(k) = (\Phi'(K^{1/2}v) \circ K^{1/2})(k) = (\Phi'(K^{1/2}v))(K^{1/2}k) = (\mathbf{f}K^{1/2}v, K^{1/2}k) = (K^{1/2}\mathbf{f}K^{1/2}v, k)$ for all $v, k \in L^2[0,1]$. So $(\Phi \circ K^{1/2})'(v) = K^{1/2}\mathbf{f}K^{1/2}v$ for all $v \in L^2[0,1]$. The proof is completed. $\square$

Theorem 3.3. *Suppose that*

$$\int_0^u f(t,v)\, dv \leqslant \frac{a}{2}u^2 + b(t)|u|^{2-\gamma} + c(t), \quad t \in [0,1],\ u \in \mathbb{R}^1, \tag{3.3}$$

where $a \in [0, \pi^2/4)$, $\gamma \in (0,2)$, $b \in L^{2/\gamma}[0,1]$, *and* $c \in L^1[0,1]$. *Then BVP* (1.1) *has at least one solution in* $C^2[0,1]$.

Proof. From Lemma 3.1 it is only necessary to prove that the operator equation $v - K^{1/2}\mathbf{f}K^{1/2}v = 0$ has at least one solution in $L^2[0,1]$.

Consider the functional $\Psi : L^2[0,1] \to \mathbb{R}^1$,

$$\Psi(v) = \frac{1}{2}(v,v) - (\Phi \circ K^{1/2})(v), \quad v \in L^2[0,1],$$

where Φ is defined in Lemma 3.2. Then, using Lemma 3.2, we have that $\Psi'(v) = v - K^{1/2}\mathbf{f}K^{1/2}v$ for all $v \in L^2[0,1]$.

It follows from Lemma 2.3 that $K^{1/2}\mathbf{f}K^{1/2} : L^2[0,1] \to L^2[0,1]$ is completely continuous, therefore Ψ is a weakly lower semi-continuous functional on $L^2[0,1]$.

For all $v \in L^2[0,1]$, it follows from (3.3) that

$$\Psi(v) = \frac{1}{2}(v,v) - \int_0^1 \int_0^{K^{1/2}v(t)} f(t,u)\, du\, dt$$
$$\geqslant \frac{1}{2}(v,v) - \frac{a}{2}(K^{1/2}v, K^{1/2}v) - \int_0^1 b(t)\left|K^{1/2}v(t)\right|^{2-\gamma} dt - \int_0^1 c(t)\, dt$$

$$\begin{aligned}
&\geqslant \frac{1}{2}\|v\|^2 - \frac{a}{2}(Kv, v) - \left(\int_0^1 \left|b(t)\right|^{2/\gamma} dt\right)^{\gamma/2} \left(\int_0^1 \left|K^{1/2}v(t)\right|^2 dt\right)^{1-\gamma/2} - c_1 \\
&= \frac{1}{2}\|v\|^2 - \frac{a}{2}(Kv, v) - b_1 (Kv, v)^{1-\gamma/2} - c_1 \\
&\geqslant \frac{1}{2}\|v\|^2 - \frac{4a}{2\pi^2}\|v\|^2 - b_1\left(\frac{2}{\pi}\right)^{2-\gamma}\|v\|^{2-\gamma} - c_1 \\
&= \frac{1}{2}(1 - 4a/\pi^2)\|v\|^2 - b_1(2/\pi)^{2-\gamma}\|v\|^{2-\gamma} - c_1,
\end{aligned}$$

where $b_1 = (\int_0^1 |b(t)|^{2/\gamma}\,dt)^{\gamma/2}$, $c_1 = \int_0^1 c(t)\,dt$.

So, $\lim_{\|v\|\to\infty} \Psi(v) = +\infty$. Therefore, according to Theorem 2.3, there exists $v_0 \in L^2[0,1]$ such that $\Psi'(v_0) = 0$, i.e., $v_0 - K^{1/2}\mathbf{f}K^{1/2}v_0 = 0$. The proof is completed. □

Example 3.1. Consider the following BVP:

$$\begin{cases} -u''(t) = au(t) + b(t)|u(t)|^p + c(t), & t \in [0,1], \\ u(0) = u'(1) = 0, \end{cases}$$

where $a \in [0, \pi^2/4)$, $p \in (0,1)$, b and c are continuous on $[0,1]$, and $c(t) \not\equiv 0$.

It is easy to verify that (3.3) is satisfied. Therefore, it follows from Theorem 3.3 that this BVP has at least one solution in $C^2[0,1]$. And it is easy to see that this solution is a nonzero solution since $c(t) \not\equiv 0$.

Theorem 3.4. *Suppose that*

(H_1) *there exist* $\mu \in [0, 1/2)$ *and* $M > 0$ *such that* $F(t,u) \triangleq \int_0^u f(t,v)\,dv \leqslant \mu u f(t,u)$ *for all* $|u| \geqslant M$ *and* $t \in [0,1]$;
(H_2) $\limsup_{u\to 0} f(t,u)/u < \pi^2/4$, $\liminf_{u\to+\infty} f(t,u)/u > \pi^2/4$ *uniformly for* $t \in [0,1]$.

Then BVP (1.1) *has at least one nonzero solution in* $C^2[0,1]$.

Proof. By (iii) of Lemma 3.1, we only need to prove that the operator equation $v = K^{1/2}\mathbf{f}K^{1/2}v$ has at least one nonzero solution in $L^2[0,1]$.

We still consider the functional Ψ defined in the proof of Theorem 3.3. We will verify that Ψ satisfies all the conditions of Theorem 2.4 (Mountain pass lemma).

First, we will prove that Ψ satisfies P.S. condition. Since $F(t,u) - \mu u f(t,u)$ is continuous on $[0,1] \times [-M, M]$, there exists $C > 0$ such that

$$F(t,u) \leqslant \mu u f(t,u) + C, \quad t \in [0,1],\ u \in [-M, M].$$

By assumption (H_1), we obtain

$$F(t,u) \leqslant \mu u f(t,u) + C, \quad t \in [0,1],\ u \in \mathbb{R}^1. \tag{3.4}$$

Let $\{v_n\} \subset L^2[0,1]$, $|\Psi(v_n)| \leqslant \beta$, $n = 1, 2, \ldots$, $\Psi'(v_n) = (I - K^{1/2}\mathbf{f}K^{1/2})v_n \to 0$. Notice that

$$\big(\Psi'(v_n), v_n\big) = (v_n - K^{1/2}\mathbf{f}K^{1/2}v_n, v_n) = \|v_n\|^2 - \int_0^1 f\big(t, K^{1/2}v_n(t)\big)K^{1/2}v_n(t)\,dt.$$

It follows from (3.4) that

$$\begin{aligned}
\beta \geqslant \Psi(v_n) &= \frac{1}{2}\|v_n\|^2 - \int_0^1 F\big(t, K^{1/2}v_n(t)\big)\,dt \\
&\geqslant \frac{1}{2}\|v_n\|^2 - \mu\int_0^1 f\big(t, K^{1/2}v_n(t)\big)K^{1/2}v_n(t)\,dt - C \\
&= \left(\frac{1}{2} - \mu\right)\|v_n\|^2 + \mu\big(\Psi'(v_n), v_n\big) - C \\
&\geqslant \left(\frac{1}{2} - \mu\right)\|v_n\|^2 - \mu\big\|\Psi'(v_n)\big\|\,\|v_n\| - C, \quad n = 1, 2, \ldots.
\end{aligned}$$

Since $\Psi'(v_n) \to 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\beta \geqslant \left(\frac{1}{2} - \mu\right)\|v_n\|^2 - \|v_n\| - C, \quad n > N_0.$$

This implies that $\{v_n\} \subset L^2[0,1]$ is bounded. Since $K^{1/2}: L^2[0,1] \to C[0,1]$ is completely continuous, $\mathbf{f}: C[0,1] \to C[0,1]$ is continuous, and $v_n - K^{1/2}\mathbf{f}K^{1/2}v_n \to 0$, we can deduce that $\{v_n\}$ has a convergent subsequence. Thus, we obtain the desired convergence property.

From $\limsup_{u\to 0} f(t,u)/u < \pi^2/4$ uniformly for $t \in [0,1]$, there exist $\varepsilon \in (0,1)$ and $\delta > 0$ such that $f(t,u) \leqslant (1-\varepsilon)(\pi^2/4)u$ for all $t \in [0,1]$ and $u \in \mathbb{R}^1$ with $u \in [0,\delta]$, and $f(t,u) \geqslant (1-\varepsilon)(\pi^2/4)u$ for all $t \in [0,1]$ and $u \in \mathbb{R}^1$ with $u \in [-\delta, 0]$. So

$$F(t,u) = \int_0^u f(t,v)\,dv \leqslant (1/8)(1-\varepsilon)\pi^2|u|^2$$
$$\text{for all } t \in [0,1] \text{ and } u \in \mathbb{R}^1 \text{ with } |u| \leqslant \delta. \tag{3.5}$$

Let $\rho = \delta$ and $\alpha = \frac{1}{2}\varepsilon\rho^2$. Then it follows from (2.11) that $\|K^{1/2}v\|_C \leqslant \|v\| = \delta$ for all $v \in \partial B_\rho$, where $B_\rho = \{v \in L^2[0,1]\colon \|v\| < \rho\}$. So by (3.5), we have

$$\begin{aligned}
\Psi(v) &= \frac{1}{2}(v,v) - \int_0^1 \int_0^{K^{1/2}v(t)} f(t,u)\,du\,dt \\
&= \frac{1}{2}(v,v) - \int_0^1 F\big(t, K^{1/2}v(t)\big)\,dt
\end{aligned}$$

$$\begin{aligned}
&\geqslant \frac{1}{2}\|v\|^2 - \frac{1}{8}(1-\varepsilon)\pi^2 \int_0^1 \bigl(K^{1/2}v(t)\bigr)^2\,dt \\
&= \frac{1}{2}\|v\|^2 - \frac{1}{8}(1-\varepsilon)\pi^2 (Kv, v) \\
&\geqslant \frac{1}{2}\|v\|^2 - \frac{1}{8}(1-\varepsilon)\pi^2 \frac{4}{\pi^2}\|v\|^2 = \frac{1}{2}\varepsilon\|v\|^2, \quad v \in \partial B_\rho.
\end{aligned} \tag{3.6}$$

This implies that $\inf_{v\in\partial B_\rho} \Psi(v) \geqslant \frac{1}{2}\varepsilon\rho^2 = \alpha > 0$.

It is obvious from the definition of Ψ and $K^{1/2}0 = 0$ that $\Psi(0) = 0$.

On the other hand, from $\liminf_{u\to+\infty} f(t,u)/u > \pi^2/4$ uniformly for $t \in [0,1]$, there exist $\varepsilon > 0$ and $\tau > 0$ such that

$$f(t,u) \geqslant \frac{\pi^2}{4}(1+\varepsilon)u, \quad u \geqslant \tau,\ t \in [0,1].$$

Since $f(t,u) - (\pi^2/4)(1+\varepsilon)u$ is continuous on $[0,1]\times[0,\tau]$, there exists $M > 0$ such that

$$f(t,u) \geqslant \frac{\pi^2}{4}(1+\varepsilon)u - M, \quad u \in [0,\tau],\ t \in [0,1].$$

Thus, the two inequalities imply that

$$f(t,u) \geqslant \frac{\pi^2}{4}(1+\varepsilon)u - M, \quad u \geqslant 0,\ t \in [0,1].$$

Therefore,

$$F(t,u) = \int_0^u f(t,v)\,dv \geqslant \frac{1}{8}\pi^2(1+\varepsilon)u^2 - Mu, \quad u \geqslant 0,\ t \in [0,1].$$

It follows from the definition of $K^{1/2}$ that $K^{1/2}e_1(t) = (2/\pi)e_1(t) = (2\sqrt{2}/\pi)\times \sin(\pi t/2) \geqslant 0$ for all $t \in [0,1]$. Thus

$$\begin{aligned}
\Psi(se_1) &= \frac{1}{2}(se_1, se_1) - \int_0^1 F\bigl(t, sK^{1/2}e_1(t)\bigr)\,dt \\
&= \frac{1}{2}s^2 - \int_0^1 F\left(t, \frac{2s}{\pi}e_1(t)\right)dt \\
&\leqslant \frac{1}{2}s^2 - \frac{\pi^2}{8}(1+\varepsilon)\int_0^1 \frac{4s^2}{\pi^2}e_1^2(t)\,dt + M\int_0^1 \frac{2s}{\pi}e_1(t)\,dt \\
&= \frac{1}{2}s^2 - \frac{1}{2}s^2(1+\varepsilon) + \frac{2}{\pi}Ms\frac{2\sqrt{2}}{\pi} \\
&= -\frac{1}{2}\varepsilon s^2 + \frac{4\sqrt{2}}{\pi^2}Ms, \quad s > 0.
\end{aligned}$$

So $\Psi(se_1) \to -\infty$ and $\|se_1\| = s \to +\infty$ as $s \to +\infty$. And then there exists a sufficiently large $s_0 > \rho$ such that $v_1 = s_0 e_1 \in L^2[0,1]$, $v_1 \notin \bar{B}_\rho$, and $\Psi(v_1) < 0$.

Thus, according to the mountain pass lemma, Ψ has a critical value $c^* > 0$, i.e., there exists $v^* \in L^2[0,1]$ such that $\Psi(v^*) = c^*$, $\Psi'(v^*) = v^* - K^{1/2}\mathbf{f}K^{1/2}v^* = 0$. It is obvious that $v^* \neq 0$ since $\Psi(0) = 0$. The proof is completed. □

Theorem 3.5. *Suppose that $f(t,u)$ is odd in u, i.e., $f(t,-u) = -f(t,u)$ for all $t \in [0,1]$ and $u \in \mathbb{R}^1$. And suppose that* (H$_1$) *in Theorem* 3.4 *is satisfied and*

$$\limsup_{u\to 0} f(t,u)/u < \pi^2/4, \qquad \liminf_{u\to+\infty} f(t,u)/u = +\infty \quad \textit{uniformly for } t \in [0,1].$$

Then BVP (1.1) *has infinitely many solutions.*

Proof. By (iii) of Lemma 3.1, we only need to prove that the operator equation $v = K^{1/2}\mathbf{f}K^{1/2}v$ has infinitely many solutions in $L^2[0,1]$. We still consider the functional Ψ defined in the proof of Theorem 3.3, and verify that Ψ satisfies all the conditions of Theorem 2.5.

In the proof of Theorem 3.4 we have proved that Ψ satisfies P.S. condition and $\Psi(0) = 0$. By the definition of Ψ and the condition $f(t,-u) = -f(t,u)$ for all $t \in [0,1]$ and $u \in \mathbb{R}^1$, it is obvious that Ψ is even. And it follows from (3.6) that Ψ satisfies the condition (i) of Theorem 2.5 for $\rho = \delta$ and $\alpha = \frac{1}{2}\varepsilon\rho^2$.

In the following, we will prove that Ψ satisfies the condition (ii) of Theorem 2.5. If this is not true, then there exists a finite dimensional subspace W of $L^2[0,1]$ and a sequence $\{v_n\} \subset W$ such that $\Psi(v_n) > 0$, $n = 1,2,\ldots$, and $\|v_n\| \to \infty$ as $n \to \infty$. Without loss of generality, we can suppose that $\|v_n\| > 0$, $n = 1,2,\ldots$. Let $t_n = \|v_n\|$, $v_n^* = t_n^{-1}v_n$, $n = 1,2,\ldots$. Then $v_n = t_n v_n^*$, $\|v_n^*\| = 1$, $n = 1,2,\ldots$, and $t_n \to +\infty$ as $n \to \infty$. Since $\{v \in W\colon \|v\| = 1\}$ is a compact subset, $\{v_n^*\}$ has a convergent subsequence, without loss of generality, let $\|v_n^* - v^*\| \to 0$ as $n \to \infty$, where $v^* \in W$ and $\|v^*\| = 1$. Since $K^{1/2}: L^2[0,1] \to L^2[0,1]$ is completely continuous, $\|K^{1/2}v_n^* - K^{1/2}v^*\| \to 0$ as $n \to \infty$. It follows from Remark 2.3 that $K^{1/2}v^* \neq 0$.

Let $u_0 = K^{1/2}v^*$, $u_n = K^{1/2}v_n^*$, $n = 1,2,\ldots$, and $a_0 = \|u_0\| > 0$. By

$$\lim\inf_{u\to+\infty} f(t,u)/u = +\infty$$

uniformly for $t \in [0,1]$, there exists $\tau > 0$ such that

$$f(t,u) \geqslant \frac{4}{a_0^2}u, \quad u \geqslant \tau,\ t \in [0,1]. \tag{3.7}$$

And since $f(t,u) - (4/a_0^2)u$ is continuous on $[0,1] \times [0,\tau]$, there exists $M \geqslant 0$ such that

$$f(t,u) \geqslant \frac{4}{a_0^2}u - M, \quad u \in [0,\tau],\ t \in [0,1]. \tag{3.8}$$

So it follows from (3.7), (3.8) that

$$f(t,u) \geqslant \frac{4}{a_0^2}u - M, \quad u \geqslant 0,\ t \in [0,1]. \tag{3.9}$$

Since $f(t,u)$ is odd in u, $F(t,u)=\int_0^u f(t,v)\,dv$ is even in u. Thus, it follows from (3.9) that

$$F(t,u)=F\big(t,|u|\big)=\int_0^{|u|} f(t,v)\,dv \geqslant \frac{2}{a_0^2}u^2-M|u|, \quad u\in\mathbb{R}^1,\ t\in[0,1].$$

Then

$$\begin{aligned}
\Psi(v_n)&=\Psi\big(t_nv_n^*\big)\\
&=\frac{1}{2}\big\|t_nv_n^*\big\|^2-\int_0^1 F\big(t,t_nK^{1/2}v_n^*(t)\big)\,dt\\
&=\frac{1}{2}\|t_nv_n^*\|^2-\int_0^1 F\big(t,t_nu_n(t)\big)\,dt\\
&\leqslant\frac{1}{2}t_n^2-\int_0^1\left(\frac{2}{a_0^2}\big|t_nu_n(t)\big|^2-M\big|t_nu_n(t)\big|\right)dt\\
&=\frac{1}{2}t_n^2-\frac{2}{a_0^2}t_n^2\|u_n\|^2+Mt_n\int_0^1\big|u_n(t)\big|\,dt\\
&\leqslant\frac{1}{2}t_n^2-\frac{2}{a_0^2}t_n^2\|u_n\|^2+Mt_n\|u_n\|, \quad n=1,2,\ldots.
\end{aligned}\tag{3.10}$$

Since $\|u_n-u_0\|\to 0$ as $n\to\infty$, $\|u_n\|\to\|u_0\|=a_0$ as $n\to\infty$. Therefore, there exists $N_0\in\mathbb{N}$ such that $\|u_n\|^2>a_0^2/2$ and $\|u_n\|\leqslant 2a_0$ as $n>N_0$. It follows from (3.10) that

$$\Psi(v_n)<\frac{1}{2}t_n^2-\frac{2}{a_0^2}\frac{1}{2}a_0^2t_n^2+2a_0Mt_n=-\frac{1}{2}t_n^2+2a_0Mt_n, \quad n>N_0.$$

Thus, noticing that $t_n\to+\infty$ as $n\to\infty$, we have that $\Psi(v_n)\to-\infty$ as $n\to\infty$. This contradicts with the assumption $\Psi(v_n)>0$, $n=1,2,\ldots$. Then according to Theorem 2.5, Ψ has infinitely many critical points, i.e., the operator equation $v=K^{1/2}\mathbf{f}K^{1/2}v$ has infinitely many solutions in $L^2[0,1]$. The proof is completed. □

Remark 3.1. If there exists $p>0$ such that

$$\lim_{u\to\infty}\frac{f(t,u)}{|u|^pu}=A>0 \quad \text{uniformly for } t\in[0,1], \tag{3.11}$$

then (H_1) holds. Let $\mu\in(1/(p+2),1/2)$. It follows from (3.11) that

$$\begin{aligned}
\lim_{u\to\infty}\frac{\int_0^u f(t,v)\,dv-\mu uf(t,u)}{|u|^pu^2}&=\lim_{u\to\infty}\left(\frac{\int_0^u f(t,v)\,dv}{|u|^pu^2}-\frac{\mu uf(t,u)}{|u|^pu^2}\right)\\
&=\lim_{u\to\infty}\left(\frac{f(t,u)}{(p+2)|u|^pu}-\mu\frac{f(t,u)}{|u|^pu}\right)
\end{aligned}$$

$$= \lim_{u\to\infty} \frac{f(t,u)}{|u|^p u}\left(\frac{1}{p+2} - \mu\right)$$
$$= A\left(\frac{1}{p+2} - \mu\right) < 0.$$

So there exists $M > 0$ such that $F(t,u) = \int_0^u f(t,v)\,dv \leqslant \mu u f(t,u)$ for all $|u| \geqslant M$ and $t \in [0,1]$.

Example 3.2. Consider the following BVP:

$$\begin{cases} -u''(t) = au(t) + b(t)\arctan u(t)\ln(1+u^2(t)) + c|u(t)|^p u(t), & t \in [0,1], \\ u(0) = u'(1) = 0, \end{cases} \tag{3.12}$$

where $a \in [0, \pi^2/4)$, $p, c > 0$, b is continuous on $[0,1]$.

It is obvious that $f(t,u) = au + b(t)\arctan u\ln(1+u^2) + c|u|^p u$, $t \in [0,1]$, $u \in \mathbb{R}^1$, is odd in $u \in \mathbb{R}^1$. By calculation, we have

$$\limsup_{u\to 0} \frac{f(t,u)}{u} = a < \pi^2/4, \qquad \liminf_{u\to+\infty} \frac{f(t,u)}{u} = +\infty, \qquad \lim_{u\to\infty} \frac{f(t,u)}{|u|^p u} = c > 0$$

uniformly for $t \in [0,1]$.

It follows from Remark 3.1 that (H_1) holds. According to Theorem 3.5, BVP (3.12) has infinitely many solutions.

References

[1] J.M. Davis, J. Henderson, P.J.Y. Wong, General Lidstone problems: Multiplicity and symmetry of solutions, J. Math. Anal. Appl. 251 (2000) 527–548.

[2] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, Berlin, 1985.

[3] L.H. Erbe, S. Hu, H. Wang, Multiple positive solutions of some boundary value problems, J. Math. Anal. Appl. 184 (1994) 640–648.

[4] L.H. Erbe, H. Wang, On the existence of positive solutions of ordinary differential equations, Proc. Amer. Math. Soc. 120 (1994) 743–748.

[5] D. Guo, Nonlinear Functional Analysis, second ed., Shandong Sci. & Tec. Press, 2001 (in Chinese).

[6] J. Henderson, H. Wang, Positive solutions for nonlinear eigenvalue problems, J. Math. Anal. Appl. 208 (1997) 252–259.

[7] K. Lan, J.R.L. Webb, Positive solutions of semilinear differential equations with sigularities, J. Differential Equations 148 (1998) 407–421.

[8] F. Li, G. Han, Generalization for Amann's and Leggett–Williams' three-solution theorems and applications, J. Math. Anal. Appl. 298 (2004) 638–654.

[9] F. Li, Y. Zhang, Multiple symmetric nonnegative solutions of second-order ordinary differential equations, Appl. Math. Lett. 17 (2004) 261–267.

[10] Z. Liu, F. Li, Multiple positive solutions of nonlinear two-point boundary value problems, J. Math. Anal. Appl. 203 (1996) 610–625.

[11] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Regional Conf. Ser. in Math., vol. 65, Amer. Math. Soc., Providence, RI, 1986.

[12] M. Struwe, Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, second ed., Springer-Verlag, Berlin, 1996.

[13] A.E. Taylor, D.C. Lay, Introduction to Functional Analysis, second ed., Wiley, 1980.

[14] E. Zeidler, Nonlinear Functional Analysis and its Applications, III: Variational Methods and Optimization, Springer-Verlag, New York, 1985.
[15] E. Zeidler, Nonlinear Functional Analysis and its Applications, I: Fixed-Point Theorems, Springer-Verlag, New York, 1986.
[16] E. Zeidler, Nonlinear Functional Analysis and its Applications, II/B: Nonlinear Monotone Operators, Springer-Verlag, New York, 1990.